Dark Matter via Massive (bi-)Gravity

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In this work we investigate the existence of relativistic models for dark matter in the context of bimetric gravity, used here to reproduce the modified Newtonian dynamics (MOND) at galactic scales. For this purpose we consider two different species of dark matter particles that separately couple to the two metrics of bigravity. These two sectors are linked together *via* an internal $U(1)$ vector field, and some effective composite metric built out of the two metrics. Among possible models only certain classes of kinetic and interaction terms are allowed without invoking ghost degrees of freedom. Along these lines we explore the number of allowed kinetic terms in the theory and point out the presence of ghosts in a previous model. Finally, we propose a promising class of ghost-free candidate theories that could provide the MOND phenomenology at galactic scales while reproducing the standard cold dark matter (CDM) model at cosmological scales.

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I. INTRODUCTION

General Relativity (GR) successfully describes the gravitational interaction in a wide range of scales and regimes, from the solar system size to strong fields in binary pulsars and black holes, and most likely will constitute the correct tool for the future gravitational wave astronomy [1]. Up to now, GR has been able to prevail against all alternative theories, either scalar-tensor [2–7], vector-tensor [8–14] or tensor-tensor theories, the latter comprising massive gravity [15, 16], bigravity [17, 18] and multigravity [19] theories.

In spite of these successes, the extrapolation of GR to a broader range of scales — notably, cosmological scales — faces important challenges since it relies on the introduction of a dark sector, composed of dark matter and dark energy. The nature of this dark sector constitutes one of the most important mystery of contemporary physics.

The reference model of cosmology today assumes a pure cosmological constant Λ added to the field equations of GR to account for the dark energy, and a component of non-baryonic dark matter made of non relativistic particles called cold dark matter (CDM). The best motivated candidate for the dark matter particle is the WIMP [20].

The model Λ-CDM is very well tested at cosmological scales by the accelerated expansion of the universe, by the observed fluctuations of the cosmic microwave background, and by the distribution of dark matter in large scale structures.

Unfortunately this model does not explain the presence of a tiny cosmological constant Λ. In the prevailing view it should be interpreted as a constant energy density of the vacuum. However, the unnatural observed value of Λ and the instability against large quantum corrections put in doubt its consistency using standard quantum field theory techniques.

Another important concern is that the model Λ-CDM does not account for many observations of dark matter at the scale of galaxies, where it faces unexplained tight correlations between dark and luminous matter in galaxy halos [21, 22]. Primary examples are the baryonic Tully-Fisher relation between the asymptotic rotation velocity of spiral galaxies and their baryonic mass, and the correlation between the mass discrepancy (*i.e.* the presence of dark matter) and the acceleration scale involved [23, 24]. These correlations happen to be very well explained by the MOND (MOdified Newtonian Dynamics) empirical formula [25–27]. The agreement between MOND and all observations at galactic scales is remarkable and calls for an explanation. On the other hand, MOND has problems explaining the DM distribution at the larger scale of galaxy clusters [28–32].

Many works have been devoted to promoting the

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MOND formula into a decent relativistic theory. Most approaches modify GR with extra fields without invoking dark matter [33–41]. Here we shall be interested in another approach, based on a form of dark matter *à la* MOND called dipolar dark matter (DDM). This approach is motivated by the dielectric analogy of MOND [42]. A first relativistic model was proposed in [43, 44] and shown to reproduce the model Λ-CDM at cosmological scales. Recently, a more sophisticated model has been based on a bimetric extension of GR [45] (see also [46] for further motivation). In this model two species of dark matter particles are coupled respectively to the two metrics, and are linked by an internal vector field generated by the mass of these particles. The phenomenology of MOND then results from a mechanism of gravitational polarization.

Bimetric theories have been extensively investigated in the quest of a consistent massive gravity theory going beyond the linear Fierz-Pauli theory. The past decade has seen the emergence of a specific theory [15, 16] that avoids the appearance of the Boulware-Deser (BD) ghost [47] to any order in perturbations. This dRGT theory [15, 16] has been extended and reformulated as a bimetric theory with two dynamical metrics [17, 18]. The theoretical and cosmological implications of these theories are extremely rich. Notably, cosmological solutions of massive gravity theories have drawn much attention [48–50] (see also the references in [51]).

In the present paper we point out that the previous model for DDM in a bimetric context [45], despite the important phenomenology it is able to reproduce, is plagued by ghosts and cannot be considered as a viable theory. Nevertheless, this phenomenology (especially at galactic scales, *i.e.* MOND) definitely calls for a more fundamental theory. We look for a consistent coupling of the dark matter fields to bigravity, closely following the restrictions made in [52–54]. We thus propose a new model, whose dark matter sector is identical to the one in the previous model [45], but whose gravitational sector is now based on ghost-free massive bigravity theory. As bigravity theory represents essentially a unique consistent deformation of GR, we think that the new model will represent an important step toward a more fundamental theory of dark matter *à la* MOND in galactic scales. In a separate paper [55] we work out in more details the new model and investigate whether it reproduces also the cosmological Λ-CDM model at large scales.

II. DIPOLAR DARK MATTER

A new relativistic model for dipolar dark matter was constructed in [45] *via* a bimetric extension of GR, which recovers successfully the phenomenology of MOND. It relies on the existence of two species for dark matter that couple to two different metrics and an additional internal field in form of a vector field,

$$
\mathcal{L} = \sqrt{-g}\left(\frac{M_g^2}{2}R_g - \rho_{\rm b} - \rho_g\right) + \sqrt{-f}\left(\frac{M_f^2}{2}R_f - \rho_f\right)
$$
$$
+ \sqrt{-\mathcal{G}_{\rm eff}}\left[M_{\rm eff}^2\left(\frac{\mathcal{R}_{\rm eff}}{2} - 2\Lambda_{\rm eff}\right)\right.
$$
$$
\left. + \mathcal{A}_\mu\left(j_g^\mu - j_f^\mu\right) + \mathcal{W}(\mathcal{X})\right], \tag{1}
$$

where $\rho_{\rm b}$, ρ_g, ρ_f are the scalar energy densities of pressureless ordinary matter (baryons) and the two species of dark matter respectively, and j_g^μ, j_f^μ denote the conserved currents of the dark matter. On top of the two Einstein-Hilbert terms for the g and f metrics, there is an additional kinetic term for the effective metric $\mathcal{G}_{\rm eff}$ and a cosmological constant $\Lambda_{\rm eff}$ associated to it (here we neglect possible cosmological constants in the g and f sectors). The $U(1)$ vector field $\mathcal{A}_\mu$ is introduced to link together the two species of dark matter particles and has a non-canonical kinetic term $\mathcal{W}(\mathcal{X})$, with

$$
\mathcal{X} = \mathcal{G}_{\rm eff}^{\mu\rho}\mathcal{G}_{\rm eff}^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\,, \tag{2}
$$

and $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$. The rich phenomenology and physical consequences of this model were studied with great detail in [45]. For a particular choice of the function $\mathcal{W}$ it recovers the desirable features of MOND and passes the constraints of the solar system. Furthermore it agrees with the cosmological model Λ-CDM at first order cosmological perturbation and is thus consistent with the fluctuations of the CMB.

The effective composite metric $\mathcal{G}_{\rm eff}$ was computed perturbatively in [45] and here we show the exact nonperturbative solution for this metric. Furthermore, we investigate the number of gravitational propagating modes and the presence of ghost instabilities. The metric $\mathcal{G}_{\mu\nu}^{\rm eff}$ was defined in [45] by the implicit relations

$$
\mathcal{G}_{\mu\nu}^{\rm eff} = \mathcal{G}_{\rm eff}^{\rho\sigma}g_{\rho\mu}f_{\nu\sigma} = \mathcal{G}_{\rm eff}^{\rho\sigma}g_{\rho\nu}f_{\mu\sigma}\,. \tag{3}
$$

After introducing the matrices $G_\mu^\nu = \mathcal{G}_{\rm eff}^{\nu\rho}g_{\mu\rho}$ and $F_\mu^\nu = \mathcal{G}_{\rm eff}^{\nu\rho}f_{\mu\rho}$ the above relations simply become

$$
GF = FG = \mathbb{1}\,, \tag{4}
$$

thus G and F are the inverse of each other. Using this fact the form of $\mathcal{G}_{\rm eff}$ can be computed. This was done perturbatively in [45] with result given by (A8) there. Actually the solution of (3)–(4) can be obtained exactly. For this, we note that the following relations are true,

$$
g^{-1}f = (\mathcal{G}_{\rm eff}G)^{-1}\,\mathcal{G}_{\rm eff}F = G^{-1}F = F^2\,. \tag{5}
$$

This means that we can identify

$$
F = \sqrt{g^{-1}f}\,. \tag{6}
$$

Thus the exact solution for the effective metric fulfilling the relations (3) is suggestively

$$
\mathcal{G}_{\mu\nu}^{\rm eff} = g_{\mu\rho}\left(\sqrt{g^{-1}f}\right)_\nu^\rho\,. \tag{7}
$$

Here we will first pay special attention to the consequences coming from the kinetic term and a cosmological constant for this effective metric $\mathcal{G}_{\text{eff}}$ in the action (1).

III. MORE ON EFFECTIVE METRICS

In the different context of massive bigravity theories, interesting proposals for an effective composite metric were made in [52–54]. There the form of the effective metric was determined by the question of how the coupling of the matter fields to the two metrics of massive bigravity behave at the quantum level and whether they alter the specific potential interactions of the allowed potential interactions between the two metrics. One particularly interesting effective composite metric has the following form,

$$g_{\mu\nu}^{\text{eff}} = \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}_{\nu} + \beta^2 f_{\mu\nu}, \quad (8)$$

where α and β are arbitrary constants. Defining the quantities $X^{\rho}_{\nu} = (\sqrt{g^{-1}f})^{\rho}_{\nu}$ and $Y_{\mu\nu} = g_{\mu\rho}X^{\rho}_{\nu}$ as was done in [52] (where $Y_{\mu\nu}$ is shown to be symmetric), it is straightforward to see that the determinant of this composite metric corresponds to the allowed potential interactions in massive bigravity,

$$\begin{aligned} \det(g_{\mu\nu}^{\text{eff}}) &= \det\left[(\alpha g_{\mu\rho} + \beta Y_{\mu\rho})\, g^{\rho\sigma}\, (\alpha g_{\nu\sigma} + \beta Y_{\nu\sigma})\right] \\ &= (\det g)^{-1}\left[\det(\alpha g_{\mu\nu} + \beta Y_{\mu\nu})\right]^2 \\ &= (\det g)\left[\det(\alpha\mathbb{1} + \beta g^{-1}Y)\right]^2. \end{aligned} \quad (9)$$

Thus, the square root of the determinant of $g_{\mu\nu}^{\text{eff}}$, say $g_{\text{eff}} = \det(g_{\mu\nu}^{\text{eff}})$, corresponds to

$$\sqrt{-g_{\text{eff}}} = \sqrt{-g}\,\det\left(\alpha\mathbb{1} + \beta X\right). \quad (10)$$

This is the right form of the acceptable potential interactions between the metrics g and f. Expanding $\sqrt{-g_{\text{eff}}}$ around a flat background, defining

$$\begin{aligned} g_{\mu\nu} &= (\eta_{\mu\nu} + h_{\mu\nu})^2, \\ f_{\mu\nu} &= (\eta_{\mu\nu} + \ell_{\mu\nu})^2, \end{aligned} \quad (11)$$

they correspond to the specific interactions of the form

$$\sqrt{-g_{\text{eff}}} = \sum_{n=0}^{4}(\alpha+\beta)^{4-n}e_n(k), \quad (12)$$

where $k_{\mu\nu} = \alpha h_{\mu\nu} + \beta\ell_{\mu\nu}$, and the symmetric polynomials are defined by (with $[\cdots]$ denoting the trace as usual)

$$\begin{aligned} e_0(k) &= 1, \\ e_1(k) &= [k], \\ e_2(k) &= \frac{1}{2}\left([k]^2 - [k^2]\right), \\ e_3(k) &= \frac{1}{6}\left([k]^3 - 3[k][k^2] + 2[k^3]\right), \\ e_4(k) &= \frac{1}{24}\left([k]^4 - 6[k]^2[k^2] + 3[k^2]^2 \right. \\ &\qquad\left. + 8[k][k^3] - 6[k^4]\right). \end{aligned} \quad (13)$$

Thus, $\sqrt{-g_{\text{eff}}}$ has exactly the nice structure of the potential with the special tuning in order to remove the BD ghost at any order [15, 16].

Finally, the relation between the effective composite metric $\mathcal{G}_{\text{eff}}$ proposed in [45] and the alternative effective metric g_{eff} proposed in [52] is given by [since $\mathcal{G}_{\mu\nu}^{\text{eff}} = Y_{\mu\nu}$ from (7)]

$$g_{\mu\nu}^{\text{eff}} = \alpha^2 g_{\mu\nu} + 2\alpha\beta\,\mathcal{G}_{\mu\nu}^{\text{eff}} + \beta^2 f_{\mu\nu}. \quad (14)$$

In other words, $\mathcal{G}_{\text{eff}} = g\sqrt{g^{-1}f}$ does not contain the linear parts proportional to g and f in g_{eff}. Unfortunately, this will have important consequences as we will see in the following section.

IV. COSMOLOGICAL CONSTANT FOR THE EFFECTIVE METRIC

We will first study the consequences of having in the model (1) the square root of the determinant $\mathcal{G}_{\text{eff}} = \det(\mathcal{G}_{\mu\nu}^{\text{eff}})$. Since our non-perturbative solution is $\mathcal{G}_{\mu\nu}^{\text{eff}} = g_{\mu\rho}(\sqrt{g^{-1}f})^{\rho}_{\nu}$, the square root of the determinant reads

$$\sqrt{-\mathcal{G}_{\text{eff}}} = \sqrt{\sqrt{-g}\sqrt{-f}}. \quad (15)$$

Perturbed around a flat background, it corresponds in the notation (11) to

$$\begin{aligned} \sqrt{-\mathcal{G}_{\text{eff}}} &= 1 + \frac{1}{2}[h+\ell] + \frac{1}{8}\left([h+\ell]^2 - 2[h^2+\ell^2]\right) \\ &\quad + \frac{1}{48}\left([h+\ell]^3 - 6[h+\ell][h^2+\ell^2] + 8[h^3+\ell^3]\right) \\ &\quad + \cdots. \end{aligned} \quad (16)$$

As is immediately seen, $\sqrt{-\mathcal{G}_{\text{eff}}}$ does not have the right potential structure, in fact it does not even contain the right structure for the linear Fierz-Pauli mass term. Any Lagrangian that contains this term as a possible potential interaction between the two metrics has immediately the BD ghost at the linear order. Thus the cosmological constant for this effective metric or any minimal coupling to matter fields *via* $\mathcal{G}_{\mu\nu}^{\text{eff}}$ will reintroduce the dangerous ghostly mode. The ghost would come already at a scale

$$m^2 M_{\text{P}}^2\sqrt{-\mathcal{G}_{\text{eff}}} \sim \frac{m^2 M_{\text{P}}^2(\Box\pi)^2}{\Lambda_3^6} = \frac{(\Box\pi)^2}{m^2}, \quad (17)$$

where $\Lambda_3^3 = M_{\text{P}}m^2$ and π denotes the 0-helicity mode. This means that the ghost is a very light degree of freedom. This immediately kills the possibility of considering any Lagrangian (independently of all the additional terms present in it) that contains $\sqrt{-\mathcal{G}_{\text{eff}}}$.

V. MINI-SUPERSPACE OF THE NEW KINETIC TERM

In the previous section we studied the implications of having the cosmological constant for $\mathcal{G}_{\mu\nu}^{\text{eff}}$ and saw that it

introduces ghostly interactions between the two metrics. In this section we will pay attention to the kinetic term $\sqrt{-\mathcal{G}_{\text{eff}}}\mathcal{R}_{\text{eff}}$, where $\mathcal{R}_{\text{eff}}$ is the Ricci scalar built from $\mathcal{G}_{\mu\nu}^{\text{eff}}$. Moreover, we will investigate the allowed number of kinetic terms. The first test that such term has to pass is the special case of the mini-superspace. The respective metrics in the mini-superspace are given by

$$\begin{aligned} \mathrm{d}s_g^2 &= g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -n_g^2\mathrm{d}t^2 + a_g^2\mathrm{d}x^2\,, \\ \mathrm{d}s_f^2 &= f_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -n_f^2\mathrm{d}t^2 + a_f^2\mathrm{d}x^2\,, \end{aligned} \quad (18)$$

where n_g, n_f and a_g, a_f are functions of the cosmic time t only. Consider the following Lagrangian with the three kinetic terms

$$\mathcal{L}_{\text{kin}}^{\text{eff}} = \frac{M_g^2}{2}\sqrt{-g}R_g + \frac{M_f^2}{2}\sqrt{-f}R_f + \frac{M_{\text{eff}}^2}{2}\sqrt{-\mathcal{G}^{\text{eff}}}\mathcal{R}_{\text{eff}}\,, \quad (19)$$

that in the mini-superspace simply becomes

$$\mathcal{L}_{\text{kin}}^{\text{eff}} = -\frac{3M_g^2 a_g \dot{a}_g^2}{n_g} - \frac{3M_f^2 a_f \dot{a}_f^2}{n_f} - \frac{3M_{\text{eff}}^2 a_{\text{eff}} \dot{a}_{\text{eff}}^2}{n_{\text{eff}}}\,. \quad (20)$$

Following the prescription (3) we obtain $n_{\text{eff}} = \sqrt{n_g n_f}$ and $a_{\text{eff}} = \sqrt{a_g a_f}$. We compute the conjugate momenta for the scale factors and get

$$\begin{aligned} p_g &= -6M_g^2 a_g^2 H_g - \frac{3}{2}M_{\text{eff}}^2 a_f \sqrt{\frac{a_g a_f}{n_g n_f}}\left(H_g n_g + H_f n_f\right)\,, \\ p_f &= -6M_f^2 a_f^2 H_f - \frac{3}{2}M_{\text{eff}}^2 a_g \sqrt{\frac{a_g a_f}{n_g n_f}}\left(H_g n_g + H_f n_f\right)\,, \end{aligned} \quad (21)$$

where $H_g = \frac{\dot{a}_g}{a_g n_g}$ and similarly H_f are the conformal Hubble factors. Now we can perform the Legendre transformation to obtain the following Hamiltonian:

$$\begin{aligned} \mathcal{H}_{\text{kin}}^{\text{eff}} &= \frac{1}{\mathcal{Q}}\Big\{a_g^2 n_f\Big(M_{\text{eff}}^2 p_g^2 \sqrt{a_g a_f}\,n_g + 4M_g^2 p_f^2 a_f \sqrt{n_g n_f}\Big) \\ &\quad + 2p_g a_g a_f n_g\Big(-M_{\text{eff}}^2 p_f \sqrt{a_g a_f}\,n_f + 2M_f^2 p_g a_f \sqrt{n_g n_f}\Big) \\ &\quad + M_{\text{eff}}^2 p_f^2 a_f^2 \sqrt{a_g a_f}\,n_g n_f\Big\}\,, \end{aligned} \quad (22)$$

where we defined the shortcut notation for convenience

$$\begin{aligned} \mathcal{Q} &= -12\Big(M_{\text{eff}}^2 M_f^2 a_f^3 \sqrt{a_g a_f}\,n_g + M_{\text{eff}}^2 M_g^2 a_g^3 \sqrt{a_g a_f}\,n_f \\ &\quad +4M_f^2 M_g^2 a_g^2 a_f^2 \sqrt{n_g n_f}\Big)\,. \end{aligned} \quad (23)$$

The Hamiltonian is highly non-linear in the lapses n_g and n_f. Since there is no shift over which we have to integrate, this is an immediate sign that these three kinetic terms have the BD ghost degree of freedom already in the mini-superspace (see [56] for an introduction to constrained hamiltonian systems). Thus, one has to avoid the two very bad contributions in form of (i) the cosmological constant term for $\mathcal{G}_{\text{eff}}$, and (ii) the kinetic term $\sqrt{-\mathcal{G}_{\text{eff}}}\mathcal{R}_{\text{eff}}$ — both these terms correspond to ghostly

interactions. Because of their very different structures there is no hope for cancellations between these terms.

Taking the limit when $M_f \to 0$ of the Hamiltonian (22) results in

$$\mathcal{H}_{\text{kin}}^{\text{eff}}\big|_{M_f\to 0} = -\frac{1}{12a_g^3}\left(\frac{(p_g a_g - p_f a_f)^2 n_g}{M_g^2} + \frac{4p_f^2 a_g \sqrt{a_g a_f}\sqrt{n_g n_f}}{M_{\text{eff}}^2}\right)\,. \quad (24)$$

As one can see, even in this limit the Hamiltonian is not linear in the lapses, so that the variation of the Hamiltonian with respect to the lapses gives rise to equations of motion that depend on the lapses and hence the constraint equation is lost. Therefore, the kinetic term $\sqrt{-\mathcal{G}_{\text{eff}}}\mathcal{R}_{\text{eff}}$ introduces ghostly interactions already in the mini-superspace independently of the number of present kinetic terms.

An interesting question to address at this stage is whether or not the mini-superspace can be made ghost-free by considering the kinetic term for g_{eff} that was proposed in [52]. Since the determinant of g_{eff} corresponds to the right ghost-free potential interactions between two metrics, the kinetic term for g_{eff} might behave better than that for $\mathcal{G}_{\text{eff}}$. Thus, consider as next the Lagrangian with the alternative three kinetic terms

$$\tilde{\mathcal{L}}_{\text{kin}}^{\text{eff}} = \frac{M_g^2}{2}\sqrt{-g}R_g + \frac{M_f^2}{2}\sqrt{-f}R_f + \frac{M_{\text{eff}}^2}{2}\sqrt{-g_{\text{eff}}}R_{\text{eff}}\,, \quad (25)$$

where R_{eff} is now the Ricci scalar of the metric $g_{\mu\nu}^{\text{eff}}$. In the mini-superspace this becomes

$$\tilde{\mathcal{L}}_{\text{kin}}^{\text{eff}} = -\frac{3M_g^2 a_g \dot{a}_g^2}{n_g} - \frac{3M_f^2 a_f \dot{a}_f^2}{n_f} - \frac{3M_{\text{eff}}^2 \tilde{a}_{\text{eff}} \dot{\tilde{a}}_{\text{eff}}^2}{\tilde{n}_{\text{eff}}}\,, \quad (26)$$

with this time $\tilde{n}_{\text{eff}} = \alpha n_g + \beta n_f$ and $\tilde{a}_{\text{eff}} = \alpha a_g + \beta a_f$. The conjugate momenta for the scale factors are now

$$\begin{aligned} \tilde{p}_g &= -6\Big(M_g^2 a_g^2 H_g + \frac{\alpha M_{\text{eff}}^2 \tilde{a}_{\text{eff}}}{\tilde{n}_{\text{eff}}}\big(\alpha a_g H_g n_g + \beta a_f H_f n_f\big)\Big)\,, \\ \tilde{p}_f &= -6\Big(M_f^2 a_f^2 H_f + \frac{\beta M_{\text{eff}}^2 \tilde{a}_{\text{eff}}}{\tilde{n}_{\text{eff}}}\big(\alpha a_g H_g n_g + \beta a_f H_f n_f\big)\Big)\,. \end{aligned} \quad (27)$$

Thus, the Hamiltonian is given by

$$\begin{aligned} \tilde{\mathcal{H}}_{\text{kin}}^{\text{eff}} &= \frac{1}{\tilde{\mathcal{Q}}}\Big\{a_g n_f\big[-\alpha(M_g^2 \tilde{p}_f^2 + M_{\text{eff}}^2(\alpha\tilde{p}_f - \beta\tilde{p}_g)^2)n_g \\ &\quad - M_g^2 \tilde{p}_f^2 \beta n_f\big] + a_f n_g\big[-M_f^2 \tilde{p}_g^2 \alpha n_g \\ &\quad - \beta(M_f^2 \tilde{p}_g^2 + M_{\text{eff}}^2(\alpha\tilde{p}_f - \beta\tilde{p}_g)^2)n_f\big]\Big\}\,, \end{aligned} \quad (28)$$

where $\tilde{\mathcal{Q}}$ stands for

$$\begin{aligned} \tilde{\mathcal{Q}} &= 12\Big(M_f^2 \alpha a_f\big[(M_g^2 + M_{\text{eff}}^2 \alpha^2)a_g + M_{\text{eff}}^2 \alpha\beta a_f\big]n_g \\ &\quad + M_g^2 \beta a_g\big[M_f^2 a_f + M_{\text{eff}}^2 \beta\tilde{a}_{\text{eff}}\big]n_f\Big)\,. \end{aligned} \quad (29)$$

Again, the Hamiltonian is highly non-linear in the lapses. The problem comes from the fact that we have too many kinetic terms. Indeed we see immediately that the only way of having linear dependence in the lapses in the mini-superspace (and hence getting rid of the BD ghost) is if we take either the $\beta \to 0$ limit — this would simply correspond to having only the standard kinetic terms for the g and f metrics — or the $M_f \to 0$ limit,

$$\tilde{\mathcal{H}}_{\text{kin}}^{\text{eff}}\big|_{M_f \to 0} = -\frac{M_{\text{eff}}^2 \beta(\alpha\tilde{p}_f - \beta\tilde{p}_g)^2 a_f n_g}{12 M_{\text{eff}}^2 M_g^2 \beta^2 a_g \tilde{a}_{\text{eff}}}$$
$$- \frac{\alpha\big(M_g^2\tilde{p}_f^2 + M_{\text{eff}}^2(\alpha\tilde{p}_f - \beta\tilde{p}_g)^2\big)n_g + M_g^2\tilde{p}_f^2\beta n_f}{12 M_{\text{eff}}^2 M_g^2 \beta^2 \tilde{a}_{\text{eff}}} \ . \quad (30)$$

As we see, the Hamiltonian becomes linear in the lapses when we remove for instance the kinetic term for the f metric. Thus, the only way of having a healthy mini-superspace is if we restrict the kinetic Lagrangian to be either $M_g^2\sqrt{-g}R_g + M_f^2\sqrt{-f}R_f$ which are the standard ghost-free kinetic terms, or $M_g^2\sqrt{-g}R_g + M_{\text{eff}}^2\sqrt{-g_{\text{eff}}}R_{\text{eff}}$. In a symmetric manner we could also remove the kinetic term for g and hence $M_f^2\sqrt{-f}R_f + M_{\text{eff}}^2\sqrt{-g_{\text{eff}}}R_{\text{eff}}$ would be also perfectly valid. In summary, one should restrict the theory to have not more than two kinetic terms in order not to reintroduce the BD ghost.

VI.   DIPOLAR DARK MATTER IN GHOST-FREE BIMETRIC THEORY

The dark matter model proposed in [45] is therefore non-viable, but nevertheless points toward an interesting connection between dark matter at small galactic scales (interpreted as DDM) and bimetric gravity. Based on our previous analysis, we would like now to propose the following new model for dipolar dark matter based on ghost-free bimetric theory,

$$\mathcal{L}_{\text{new}} = \sqrt{-g}\bigg(\frac{M_g^2}{2}R_g - \rho_{\text{b}} - \rho_g\bigg) + \sqrt{-f}\bigg(\frac{M_f^2}{2}R_f - \rho_f\bigg)$$
$$+ \sqrt{-g_{\text{eff}}}\bigg[m^2 M_{\text{eff}}^2 + \mathcal{A}_\mu\big(j_g^\mu - j_f^\mu\big) + \mathcal{W}(X)\bigg] , \quad (31)$$

where the ghost-free potential interactions are defined by the metric (8) [they take the form (12)–(13) when expanded around a flat background], and where the kinetic term of the vector field is now constructed with the metric $g_{\mu\nu}^{\text{eff}}$,

$$X = g_{\text{eff}}^{\mu\rho}g_{\text{eff}}^{\nu\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} \ . \quad (32)$$

As was shown in [52–54], the matter fields can separately couple to either the g metric or f metric without invoking the BD ghost. Additionally the matter fields can couple to the effective composite metric g_{eff} which is ghost-free in the mini-superspace and in the decoupling limit.

Here we propose to couple the ordinary baryonic fields with mass density ρ_{b} to the standard g metric while coupling the two species of dark matter with densities ρ_g

and ρ_f separately to the g and f metrics respectively. Furthermore, in order to link together the two species of dark matter particles, we consider a vector field $\mathcal{A}_\mu$ that minimally couples to the effective metric g_{eff}. The vector field plays the role of a "graviphoton" since it is generated by the mass currents j_g^μ and j_f^μ of the particles. The presence of this internal field is necessary to stabilize the dipolar medium and is expected to yield the wanted mechanism of polarization.

The model (31) fulfils the restrictions coming from our previous analysis, as it contains no more than two kinetic terms (in particular the problematic kinetic term for $\mathcal{G}_{\text{eff}}$ is absent), and the potential interactions between the two metrics coming from the square root of the determinant of g_{eff} correspond to the ghost-free prescriptions.

However, one needs also to be careful with the assumptions on the dark matter fields and their currents. Indeed the ghost could still be present in the matter sector.[1] Since the dark matter fluids that live on the g and f metrics directly couple to A_μ that lives on the g_{eff} metric, there is a priori the danger of having the ghost present due to the interaction term in the matter sector. We will investigate this question with great detail in [55] and see if for a specific choice of the dark matter fields the ghost can be maintained absent. For this we shall perform the decoupling limit analysis of our new model (31) including the matter sector and study the required amount of initial conditions.

Finally, the model (31) should share the nice properties and phenomenology of the model proposed in [45] and therefore provides a promising road for a relativistic dipolar dark matter model to be investigated. The MOND phenomenology, the PPN parameters and the cosmology of this model will be studied in a separate paper [55].

VII.   CONCLUSIONS

We explored the possible candidates for relativistic dark matter models in bimetric extensions of General Relativity, that hopefully will provide modified Newtonian dynamics (MOND) at galactic scales while giving rise to an expansion at cosmological scales. A promising road comes from the ghost-free constructions of dRGT massive gravity [15, 16] where the interactions between two metrics are tuned in a way that the Boulware-Deser ghost remains absent. Furthermore, the important studies of possible consistent couplings to matter fields [52–54] are beneficial to us, since for the model to work, we have to consider two different species of dark matter particles that couple separately to the two metrics while an additional internal vector field couples minimally to an effective metric built out of the two. The vector field

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links together the two sectors of the dark matter particles and plays a crucial role for gravitational polarization and MOND [45, 46].

For the ghost absence the question of allowed kinetic interactions is mandatory. We showed that the kinetic Lagrangian containing three kinetic terms immediately gives rise to the introduction of the ghost and we therefore concluded that only two kinetic terms are allowed.

In a future work [55], we will study in detail the covariant equations of motion of the new model, derive the non-relativistic limit and see if the polarization mechanism for dark matter works in the same way as in the originally proposed model. We will investigate in detail the possible danger of ghostly interactions in the matter sector and constrain further the model. We intend also to check if the parametrized post-Newtonian parameters are close to the ones of GR in the solar system, and to investigate the cosmological solutions in first order perturbations.

Acknowledgments

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